

# Gravitating $\sigma$ Model Solitons

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## Abstract

We study axially symmetric static solitons of  $O(3)$  nonlinear  $\sigma$  model coupled to (2+1)-dimensional anti-de Sitter gravity. The obtained solutions are not self-dual under static metric. The usual regular topological lump solution cannot form a black hole even though the scale of symmetry breaking is increased. There exist nontopological solitons of half integral winding in a given model, and the corresponding spacetimes involve charged Bañados-Teitelboim-Zanelli black holes without non-Abelian scalar hair.

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## I. INTRODUCTION

Three-dimensional (3D) Einstein gravity is characterized by the absence of propagating gravitational degree [1]. Though it is different from the nature of (3+1)-dimensional gravity, 3D gravity without the graviton has attracted attention in cosmology in connection with cosmic strings [2] and in gauge theory formulation [3]. In both contexts, (2+1)-dimensional [(2+1)D] anti-de Sitter gravity may be intriguing because it was the first example reformulated as a Chern-Simons gauge theory of the Poincaré group [3] and its vacuum solutions support black holes [4].

(2+1)D gravity with a nonzero cosmological constant was first studied in Ref. [5]. When a static point particle with mass and without spin is coupled to gravity, general anti-de Sitter solution was obtained

$$ds^2 = \sqrt{\varepsilon} \frac{(\frac{R}{R_0})^{\sqrt{\varepsilon} c} + (\frac{R_0}{R})^{\sqrt{\varepsilon} c}}{(\frac{R}{R_0})^{\sqrt{\varepsilon} c} - (\frac{R_0}{R})^{\sqrt{\varepsilon} c}} dt^2 - \frac{4\varepsilon c^2 (dR^2 + R^2 d\Theta^2)}{|\Lambda| R^2 \left[ (\frac{R}{R_0})^{\sqrt{\varepsilon} c} - (\frac{R_0}{R})^{\sqrt{\varepsilon} c} \right]^2}, \quad (1.1)$$

where  $c = 1 - 4Gm$  and  $\varepsilon$  is  $\pm 1$  for the negative cosmological constant  $\Lambda$ . When  $\varepsilon = +1$ , the metric (1.1) describes a hyperboloid with deficit angle. Note that the effect of the point particle at the origin appears only in the deficit angle in Eq. (1.1), and thereby these solutions go to vacuum solutions in the massless limit ( $m \rightarrow 0$ ). Later the Banãdos-Teitelboim-Zanelli (BTZ) black hole solutions were reported in Ref. [4], and the simplest one is the Schwarzschild-type black hole

$$ds^2 = (|\Lambda| r^2 - 8GM) dt^2 - \frac{dr^2}{|\Lambda| r^2 - 8GM} - r^2 d\theta^2. \quad (1.2)$$

Here an integration constant  $M$  of Einstein equation is arbitrary, however solutions of positive  $M$  correspond to the BTZ black holes. Since both solutions in Eqs. (1.1) and (1.2) are vacuum solutions in the limit of zero point particle mass, one may easily find a coordinate transformation to connect the  $m = 0$  solutions in Eq. (1.1) with the solutions in Eq. (1.2). As expected,  $\varepsilon = +1$  case in Eq. (1.1) corresponds to the negative  $M$  solution in Eq. (1.2), and the corresponding space is a regular hyperboloid.  $\varepsilon = -1$  case results in the exterior region of the Schwarzschild-type BTZ black hole [6].

This BTZ black hole has so far attracted much interest in various classical black hole solutions [7], in thermodynamic and statistical properties [8,9], and in string related topics [10]. In 3+1 dimensions, gravitating solitons and sphalerons have received considerable impetus by the discovery of a class of non-Abelian black hole solutions [11–13]. It might be an intriguing direction to ask the same question that whether or not gravitating solitons in (2+1)D anti-de Sitter spacetime can form solitonic BTZ black holes. In case of global U(1) vortices, a regular configuration could make a black hole structure with two horizons similar to the charged BTZ black hole [6]. Since the energy of a static global U(1) vortex diverges logarithmically in flat spacetime, we here want to address the same question to a model containing finite energy soliton excitations. In this context O(3) nonlinear  $\sigma$  model may be an appropriate choice since the field content of the model is simple, and exact static self-dual multi-soliton solutions of finite energy have been obtained in both flat and curved spacetime with zero cosmological constant [14–16].

In this paper, we consider both negative cosmological constant and matter distribution provided by regular static solitons of O(3) nonlinear  $\sigma$  model. The metric of our consideration is static and axially symmetric. The inclusion of a negative cosmological constant makes us expect to induce drastic change to solitonic physics in 2+1 dimensions. A role of it is effectively equivalent to the introduction of angular momentum under a stationary metric, and then the corresponding spacetime provides a rotating frame to the test particle. Therefore, static  $\sigma$  solitons in anti-de Sitter spacetime cannot remain to be self-dual under the static metric. Even if we obtain the self-dual  $\sigma$  solitons under the stationary metric, we encounter unphysical situation, e.g., closed timelike curves [17]. Attractive gravitational force sounds natural in 3+1 dimensions for localized ordinary matter distributions, so that it makes the matter collapse into the black hole or coagulates a new localized object which does not exist in flat spacetime [11]. Since (2+1)D gravity itself does not contain propagating gravitational field, negative vacuum energy can induce a similar effect in curved spacetime. In O(3) nonlinear  $\sigma$  model, we present a new nontopological soliton solution of half integral winding in addition to the well-known topological lump solution of integral

winding. We also show that any regular topological lump whose energy is localized near its core cannot form spacetime of a BTZ black hole. However, the nontopological solutions have a logarithmically divergent energy tail, so that their spacetimes can include charged BTZ black hole. In these aspects the obtained nontopological solitons resemble global U(1) vortices, but the non-Abelian scalar hair of  $\sigma$  solitons do not penetrate the horizon while the scalar hair of the global U(1) vortices can be observed outside the BTZ black hole.

This paper is organized as follows. In section II, we introduce the model and obtain all possible static regular solitons with axial symmetry by solving second order Euler-Lagrange equations. In section III, the spacetime structure including BTZ black holes is analyzed for the obtained gravitating solitons. Geodesic motions are computed in Sec. IV. We conclude in Sec. V with a discussion.

## II. MODEL AND SOLITON SOLUTIONS

Nonlinear  $\sigma$  model with O(3) symmetry is described by the Lagrange density

$$\mathcal{L} = -\frac{1}{16\pi G}(R + 2\Lambda) + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^a - \frac{\lambda(x)}{2}v^2(\phi^a\phi^a - v^2), \quad (2.1)$$

where a Lagrange multiplier  $\lambda(x)$  is rescaled to a dimensionless quantity, and the variation of it produces a constraint for the scalar field:  $\phi^a\phi^a = v^2$  ( $a = 1, 2, 3$ ). Throughout this paper, the dimension counting of fields is adjusted to that in (3+1)-dimensional spacetime since we presume to apply the obtained results to the straight, infinite strings. Then the model involves three mass scales, namely the Planck scale  $1/\sqrt{G}$ , the scale of negative cosmological vacuum energy  $\sqrt{|\Lambda|}$ , and the symmetry breaking scale  $v$ . Solitonic objects of our interest have axial symmetry, i.e., the corresponding string spacetime is invariant under the rotation to, and the translation along a symmetry axis. The mass in this paper stands for mass per unit length along the symmetry axis. In this case the static metric of this spacetime can be parametrized as

$$ds^2 = e^{2N(r)}B(r)dt^2 - \frac{dr^2}{B(r)} - r^2d\theta^2 - dz^2. \quad (2.2)$$

For this kind of the metric all physical settings are effectively reduced the hypersurface orthogonal to the symmetry axis, and the string-like object can be viewed as a point-like source in 2+1 dimensions. Suppose that a given matter distribution is specialized to the case of axially symmetric time-independent fields and the equations of motions are solved. The resulting metric has two integration constants that are identified as the mass and angular momentum [4]. Since we take a static metric (2.2) here, it is equivalent to set the angular momentum zero. When we fix the boundary condition at the origin for the fields and the metric, we will choose a value of the mass parameter  $B(0)$  later. We take a stereographic projection for  $\phi^a$  so that the ansatz for the solitons with axial symmetry is

$$\phi^a = v(\sin F(r) \cos n\theta, \sin F(r) \sin n\theta, \cos F(r)). \quad (2.3)$$

Euler-Lagrange equations derived from the action and the static metric are

$$\frac{d^2 F}{dr^2} + \left( \frac{dN}{dr} + \frac{1}{B} \frac{dB}{dr} + \frac{1}{r} \right) \frac{dF}{dr} = \frac{n^2}{Br^2} \sin F \cos F, \quad (2.4)$$

$$\frac{1}{r} \frac{dN}{dr} = 8\pi G v^2 \left( \frac{dF}{dr} \right)^2, \quad (2.5)$$

$$\frac{1}{r} \frac{dB}{dr} = 2|\Lambda| - 8\pi G v^2 \left\{ B \left( \frac{dF}{dr} \right)^2 + \frac{n^2}{r^2} \sin^2 F \right\}. \quad (2.6)$$

A physical condition for spacetime manifold is the reproduction of Minkowski spacetime in the limit of no matter ( $T^\mu_\nu = 0$ ) and zero cosmological constant ( $\Lambda = 0$ ), and then an appropriate set of boundary conditions is

$$B(0) = 1 \quad \text{and} \quad N(\infty) = 0. \quad (2.7)$$

When  $n \neq 0$ , well-definedness of the scalar field  $\phi^a$  in Eq. (2.3) forces the boundary condition at the origin such as

$$F(0) = 0 \quad (\text{or} \quad \sin F(0) = 0). \quad (2.8)$$

Introducing a new variable  $\tilde{r} = \ln r$  ( $-\infty < \tilde{r} < \infty$ ), we rewrite Eq. (2.4) such as

$$\frac{d^2 F}{d\tilde{r}^2} + \left( \frac{dN}{d\tilde{r}} + \frac{1}{B} \frac{dB}{d\tilde{r}} \right) \frac{dF}{d\tilde{r}} = \frac{n^2}{B} \sin F \cos F. \quad (2.9)$$

After eliminating derivative terms of the metric functions by use of Eqs. (2.5) and (2.6), we obtain

$$B \frac{d^2 F}{d\tilde{r}^2} = n^2 \sin F \cos F - (2|\Lambda|e^{2\tilde{r}} - 8\pi G v^2 n^2 \sin^2 F) \frac{dF}{d\tilde{r}}. \quad (2.10)$$

From the vanishment of the right-hand side of Eq. (2.9) at spatial infinity, we read possible boundary values of the scalar amplitude:

$$F(\infty) = \begin{cases} \pi & \text{from the sine term,} \\ \pi/2 & \text{from the cosine term,} \\ \alpha \ (0 < \alpha \leq \pi) & \text{from } 1/B(\infty) \text{ term.} \end{cases} \quad (2.11)$$

The boundary condition in the last line of Eq. (2.11) comes from the divergence of  $B(r)$  at spatial infinity. Precisely,  $B(r) \approx |\Lambda|r^2$  for a sufficiently large  $r$ .

Before analyzing  $n \neq 0$  solutions of Eq. (2.4), we will show that there does not exist  $n = 0$  regular nontrivial solution of this equation even in anti-de Sitter space. If we substitute Eqs. (2.5) and (2.6) into Eq. (2.4) when  $n = 0$ , we obtain

$$\frac{d^2 F}{dr^2} + \left( \frac{2\Lambda r}{B} + \frac{1}{r} \right) \frac{dF}{dr} = 0. \quad (2.12)$$

Since  $B(0) = 1$ ,  $F$  given by a solution of this equation contains a logarithmic divergence at the origin, i.e.,  $F(r) \propto \int dr^2 e^{-|\Lambda|r^2}/r^2$  for a sufficiently small  $r$ . Now that we have shown nonexistence of the  $n = 0$  solution, let us look for the  $n \neq 0$  soliton solutions of the equations (2.4), (2.5), and (2.6) satisfying the boundary conditions in Eqs. (2.7), (2.8), and (2.11).

## II.1 Topological Soliton

Solutions satisfying the boundary condition that  $F(0) = 0$  and  $F(\infty) = \pi$  are topological solitons when the base spatial manifold formed by them is topologically equivalent to two dimensional Euclidean space. These static solitons are characterized by topological charge,

$$Q = \frac{1}{8\pi} \int d^2x \epsilon^{0ij} \epsilon^{abc} \phi^a \partial_i \phi^b \partial_j \phi^c, \quad (2.13)$$

$$= \frac{n}{2} (\cos F(0) - \cos F(\infty)), \quad (2.14)$$

$$= n, \quad (2.15)$$

and this quantized charge  $n$  represents a winding number of second homotopy group, that is,  $\Pi_2(S^2) = Z$ . From now on we will call topological solitons of this model ‘topological lumps’.

The topological lumps are known to be unique static soliton species of  $O(3)$  nonlinear  $\sigma$  model in flat spacetime, and they have been studied in curved spacetime as a candidate of global cosmic strings [14–16]. Since the exact soliton solutions were obtained by solving first order self-dual equation, their existence has been automatic as far as the cosmological constant has not been taken into account. As we shall discuss it later, static solitons under the static metric are not self-dual in anti-de Sitter spacetime and then we have to consider the second order Euler-Lagrange equation (2.4) directly.

Since we cannot exactly solve the equations (2.4), (2.5), and (2.6), let us attempt series expansion of the fields near the origin

$$F(r) \approx F_0 r^n - (|\Lambda| - 8\pi G v^2 F_0^2 + F_0^3/2) r^3, \quad (2.16)$$

$$N(r) \approx N_0 + 4\pi G v^2 F_0^2 n r^{2n}, \quad (2.17)$$

$$B(r) \approx 1 + \left[ \frac{|\Lambda|}{v^2} - 4\pi G(1 + n^2) F_0^2 \delta_{1,n} \right] (vr)^2, \quad (2.18)$$

where  $F_0$  and  $N_0$  are constants determined by proper behavior of the fields at asymptotic region. For large  $r$  the leading term approximation gives

$$F(r) \approx \pi - \frac{F_\infty}{r^2}, \quad (2.19)$$

$$N(r) \approx -\frac{8\pi G v^2 F_\infty^2}{r^4}, \quad (2.20)$$

$$B(r) \approx |\Lambda| r^2 + B_\infty + \frac{16\pi G v^2 |\Lambda| F_\infty^2}{r^2}, \quad (2.21)$$

where  $F_\infty$  and  $B_\infty$  are also determined by the proper functional behavior at the origin.

# FIGURES

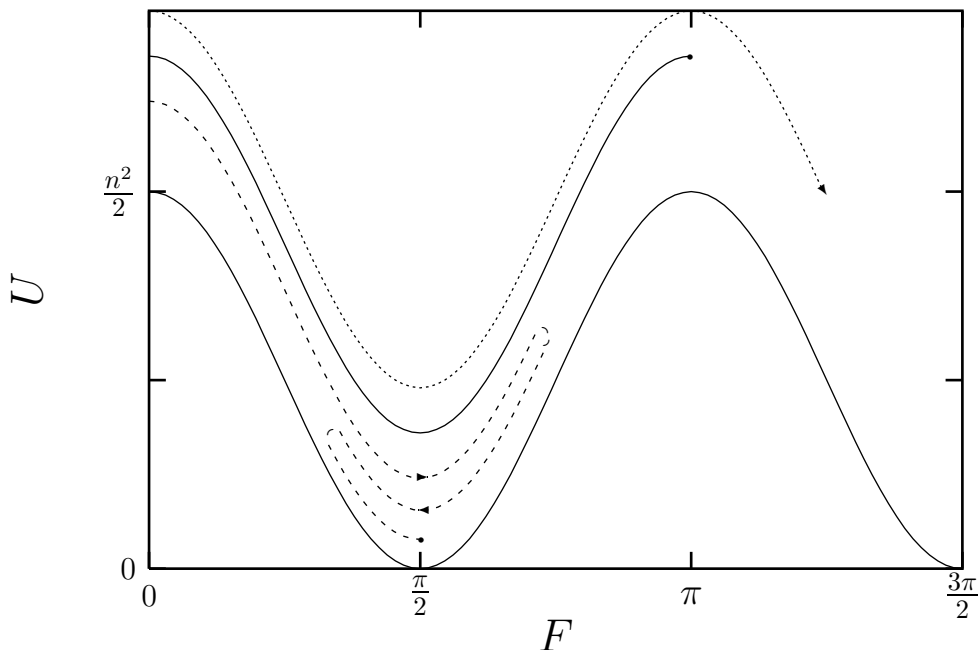


FIG. 1. Shape of the effective potential  $U$  and possible motions of a hypothetical particle: (a) overshoot solution (the dotted line), (b) critical solution with  $F(\infty) = \pi$  (the solid line), (c) undershoot solution with  $F(\infty) = \pi/2$  (the dashed line).

If we identify  $F$  as a coordinate and  $\tilde{r}$  as time in Eq. (2.10), then we can interpret this equation as a Newtonian equation for one-dimensional motion of a hypothetical particle with variable mass  $B(r)$ . The exerted forces are friction or a kind of velocity-dependent force proportional to  $dF/d\tilde{r}$ , and the conservative force from the potential  $U = \frac{n^2}{2} \cos 2F$  (See Fig. 1).

If we naively read possible motions of a hypothetical particle from the potential  $U(F)$ , then the motions satisfying  $F(r=0) = 0$  are classified into three sets by its initial velocity which can actually be replaced by the value of  $F_0$  in Eq. (2.16). When  $F_0$  is larger than a critical value, the particle reaches  $\pi$  at a finite time  $\tilde{r}$  and it corresponds to an overshoot shown by the dotted line in Fig. 1. When  $F_0$  is smaller than the critical value, the particle cannot reach  $\pi$  because of the power loss due to the velocity-dependent terms in Eq. (2.10) and this motion should have a turning point between  $\pi/2$  and  $\pi$ . The existence of the



overshoot solution given by the dotted line in Fig. 1 and the undershoot solution given by the dashed line in Fig. 1 guarantees, by continuity argument, the existence of the topological lump solution connecting  $F(r = 0) = 0$  and  $F(r = \infty) = \pi$  smoothly (See the solid line in Fig. 1).

For the metric functions,  $N(r)$  is monotonically increasing since the right-hand side of Eq. (2.5) is always nonnegative, however  $N(r)$  is slowly varying function in the asymptotic region as was shown in Eq. (2.20). It means that the exponential of  $N(r)$  does not affect much to the structure of spacetime. On the other hand, functional behavior of  $B(r)$  changes drastically according to both the magnitude of the cosmological constant and the matter distribution. Therefore, its spacetime structure, e.g., a black hole, is determined by reading the shape of  $B(r)$ . We will discuss possible spacetimes generated by various  $\sigma$  solitons in the next section.

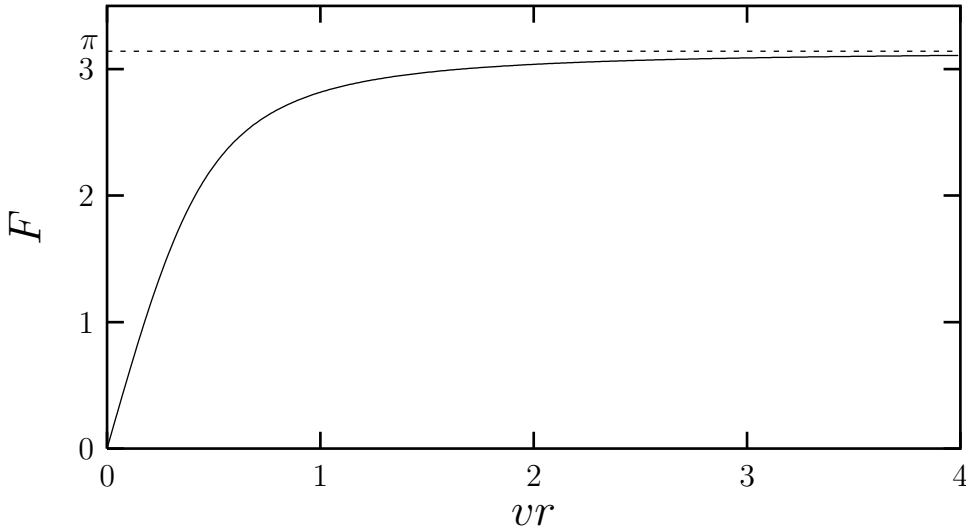


FIG. 2. A configuration of topological lump solution when  $8\pi Gv^2 = 0.2$ ,  $|\Lambda|/v^2 = 4.0 \times 10^{-6}$ , and  $F_0 = 5.896$ . The boundary value of the topological lump solution has  $\pi$  with  $10^{-6}$  precision.

In the above discussion, we neglected the effect of the variable mass  $B(r)$  in Eq. (2.10). It may be valid when the absolute value of the cosmological constant is small. On the other hand, if  $|\Lambda|/v^2$  is large enough, the terms proportional to the cosmological constant dominate even for some finite  $\tilde{r}$  region. In the Newtonian equation (2.10), such terms are

interpreted as the variable mass term  $B(\tilde{r}) \sim |\Lambda|e^{2\tilde{r}}$  and the time-dependent coefficient of the friction  $2|\Lambda|e^{2\tilde{r}}$  in the right-hand side of Eq. (2.10), respectively. In this case, the mass of the hypothetical particle can rapidly increase for small  $r$  and it can forbid the existence of overshoot solutions even for huge  $F_0$  values. It is indeed the case which was confirmed by numerical computation. In synthesis, there exists regular topological lump solution satisfying the boundary conditions,  $F(0) = 0$  and  $F(r = \infty) = \pi$ , only when  $|\Lambda|/v^2$  is less than a critical value. An example of the topological lump is shown in Fig. 2.

## II.2 Nontopological Soliton

When we discussed solutions of Eq. (2.11) in the previous subsection, we discussed possibility of another set of regular solution satisfying  $F(\infty) = \alpha$  ( $0 < \alpha < \pi$ ) as given in Eq. (2.11). Suppose that there exist such solutions and we attempt power series expansion of them for large  $r$ :

$$F(r) \sim \alpha - \frac{F_{\alpha,\infty}}{r^q}. \quad (2.22)$$

From Eqs. (2.5) and (2.6), we have

$$N(r) \sim -4\pi G v^2 q \frac{F_{\alpha,\infty}^2}{r^{2q}}, \quad (2.23)$$

$$B(r) \sim |\Lambda|r^2 + 1 - 8G\mathcal{M}_\alpha - 8\pi G v^2 n^2 \sin^2 \alpha \ln r/r_{\tilde{c}}, \quad (2.24)$$

where  $F_{\alpha,\infty}$  and  $\mathcal{M}_\alpha$  are constants which have to be chosen by the proper behavior of  $F(r)$  and  $B(r)$  near the origin, and  $r_{\tilde{c}}$  stands for core radius. Inserting the series solutions (2.22), (2.23), and (2.24) into the equation (2.4) of the scalar field, we have a relation for the leading term

$$-q(q-2)\frac{|\Lambda|F_{\alpha,\infty}}{r^q} = \frac{n^2}{r^2} \sin \alpha \cos \alpha. \quad (2.25)$$

When  $\alpha \neq \pi/2$  and  $0 < \alpha < \pi$ , the functional behavior of the radial coordinate forces  $q = 2$  but then the equality cannot hold because of the vanishment of the left-hand side of

Eq. (2.25). This implies impossibility of regular  $F(\infty) = \alpha$  solution except  $F(\infty) = \pi/2$  solution. When the boundary value of  $F$  is  $\pi/2$ , the charge defined in Eq. (2.13) is a multiple of half, i.e.,  $Q = n/2$ . Therefore, every solution of  $F(\infty) = \pi/2$  is classified as a static nontopological soliton of half integral winding.

In the previous subsection we mentioned existence of undershoot solutions, and they should be nothing but the solutions of  $F(\infty) = \pi/2$ . Here let us emphasize again the impossibility of this half integral winding solution in flat spacetime. Since  $N(r) = 0$  and  $B(r) = 1$  in flat spacetime, Eq. (2.9) depicts a one-dimensional motion of a hypothetical particle with unit mass of which position is  $F$  at time  $\tilde{r}$ . The exerted force comes only from the conservative potential  $U(F)$  shown in Fig. 1, so virial theorem allows two regular solutions, i.e., the stopped motion ( $F(\tilde{r}) = 0$ ) or the motion satisfying  $F(\tilde{r} = -\infty) = 0$  and  $F(\tilde{r} = \infty) = \pi$ . In curved spacetime with zero cosmological constant, the velocity-dependent force is not a friction but it pushes the hypothetical particle outward. Moreover the variable mass  $B(r)$  of the particle decreases as time  $\tilde{r}$  elapses. These two factors make turning of the hypothetical particle more difficult before  $F = \pi$  and forbid undershoot solution. Therefore, there does not exist any nontopological solitons of half integral winding in curved spacetime when the cosmological constant vanishes. In de Sitter spacetime, the positive cosmological constant term makes the situation worse, so we easily expect no half integral winding solution similar to the case of zero cosmological constant. In anti de Sitter spacetime, the negative cosmological constant term provides a friction as shown in Eq. (2.10) and lets the variable mass  $B(r)$  get heavy for large  $r$  as given in Eq. (2.21). Among the solutions classified by the value of  $F_0$  in Eq. (2.16), a set of  $F_0$ 's less than the critical value for the topological lump solution provides a set of undershoot solutions with turning point between  $\pi/2$  and  $\pi$ . Since the potential  $U$  has minimum at  $\pi/2$ , it may oscillate around  $\pi/2$  and finally converges to  $\pi/2$  due to the friction.

For better understanding of the asymptotic behavior of the scalar field  $F(r)$ , let us consider linearized equation for  $\delta F(r)$  defined by  $F(r) = \pi/2 + \delta F(r)$ . As an approximation of  $B(r)$  we bring up two cases: One describes the region of slowly varying  $B$  ( $B(r) \approx \bar{B}$ ),

and the other is the asymptotic region ( $B(r) \approx |\Lambda|r^2$ ). The former leads to

$$\bar{B} \frac{d^2 \delta F}{dr^2} + 3|\Lambda|r \frac{d\delta F}{dr} + \frac{n^2}{r^2} \delta F = 0, \quad (2.26)$$

and the latter goes to

$$|\Lambda|r^2 \frac{d^2 \delta F}{dr^2} + 3|\Lambda|r \frac{d\delta F}{dr} + \frac{n^2}{r^2} \delta F = 0. \quad (2.27)$$

A representative asymptotic solution of each equation is given in Fig. 3 and every solution includes both oscillation and damping as expected. Note that oscillations are rapid for small  $r$  but the period of each oscillation also increases rapidly as  $r$  increases. Since this small  $r$  region of rapid oscillation is covered by the soliton core, we may expect possibility of monotonic solution. It is indeed a case and we obtain a class of solutions specified by the number of  $\pi/2$  points at finite  $r$ . From now on we will call this number as “node”. From the value of  $F_0$  in Fig. 4 one may easily read proportionality between  $F_0$  and the nodes. Obviously the maximum value of  $F$  also increases as  $F_0$  becomes larger.

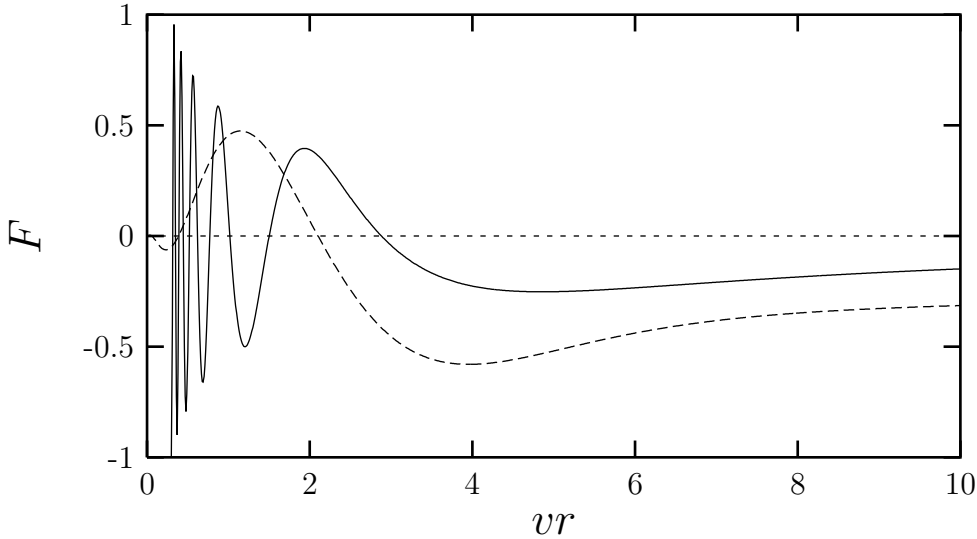


FIG. 3. Two types of asymptotic solutions for  $\delta F(r) \equiv F(r) - \pi$  when  $8\pi Gv^2 = 0.4$  and  $|\Lambda|/v^2 = 0.01$ . Dashed line is a solution of Eq. (2.26) when  $F_0 = 0.15$ , and  $F(r = 0.01) = 0.0001$ . Solid line is a solution of Eq. (2.27) when  $F_0 = 10$ , and  $F(r = 0.3) = -1$ .

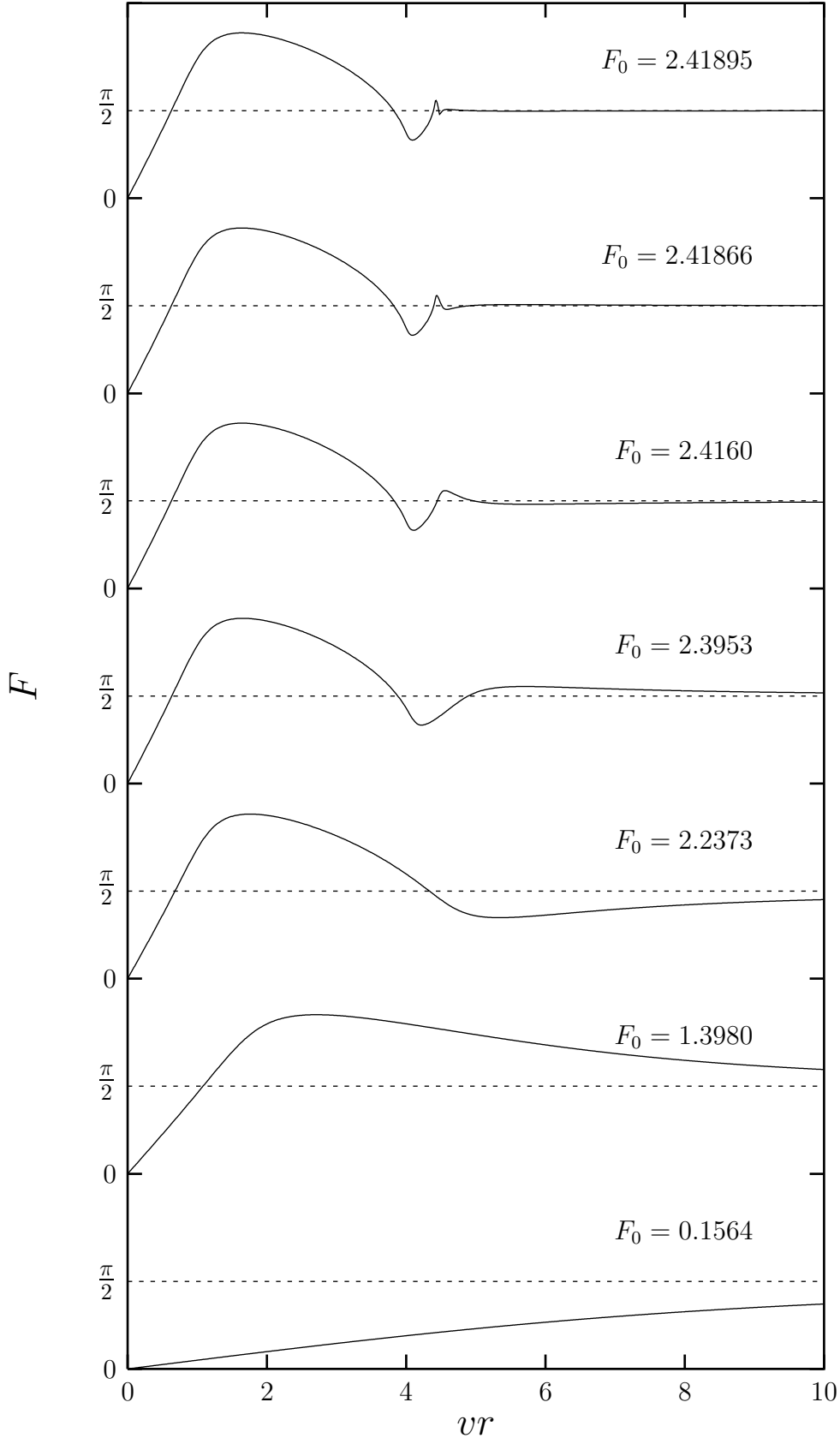


FIG. 4. Various nontopological solitons specified by the number of nodes when  $8\pi Gv^2 = 0.4$  and  $|\Lambda|/v^2 = 0.01$ .

Now some comments on  $B(r)$  for large  $r$  are in order. The expression (2.24) involves logarithmic term when  $\alpha = \pi/2$ , and it means resemblance between the obtained nontopological solitons of half integral winding and the vortices in a scalar model with global U(1) symmetry [6]. Appearance of this logarithmic term also implies that the coordinate  $r$  may not be a good coordinate for the expansion of  $B(r)$  in asymptotic region as have been done in the global U(1) vortices [18,19].

It is well-known that O(3) nonlinear  $\sigma$  model in (2+1)D flat spacetime supports self-dual solitons described by the first-order equation

$$\partial_i \phi^a = \pm \frac{1}{v} \epsilon_i^j \epsilon^{abc} \phi^b \partial_j \phi^c, \quad (2.28)$$

and any static regular topological soliton with finite energy satisfying Euler-Lagrange equation is proved to be self-dual and to satisfy Eq. (2.28). Here it would be natural to ask a question whether or not the obtained solutions in anti-de Sitter space are self-dual. In curved spacetime, second-order equation from the self-dual equation (2.28) is

$$\nabla^2 \phi^a - \frac{1}{v^2} (\phi^b \nabla^2 \phi^b) \phi^a = \pm \frac{1}{v} \epsilon^{abc} (\partial_j \epsilon^{ji} + \Gamma_{jk}^j \epsilon^{ki}) \phi^b \partial_i \phi^c, \quad (2.29)$$

where  $\nabla^2$  denotes two-dimensional Laplacian. In the static metric (2.2), Eq. (2.29) becomes

$$\begin{aligned} B \frac{d^2 F}{dr^2} + \left( B \frac{dN}{dr} + \frac{dB}{dr} + \frac{B}{r} \right) \frac{dF}{dr} - \frac{n^2}{r^2} \sin F \cos F \\ = \pm \frac{1}{v} \frac{e^N}{r} \left( B \frac{dN}{dr} + \frac{1}{2} \frac{dB}{dr} \right) n \sin F. \end{aligned} \quad (2.30)$$

Comparing Eq. (2.30) with the Euler-Lagrange equation (2.4), we obtain a necessary condition for the metric, that is, the vanishment of the right-hand side of Eq. (2.30):

$$\frac{dN}{dr} + \frac{1}{2B} \frac{dB}{dr} = 0. \quad (2.31)$$

The solution of Eq. (2.31) with a rescaling of time coordinate leads to

$$ds^2 = dt^2 - dz^2 - \frac{dr^2}{B(r)} - r^2 d\theta^2. \quad (2.32)$$

It is the very metric admitting self-dual string-like solutions in curved spacetime with zero cosmological constant [14,15]. With the help of Eq. (2.31), Eqs. (2.5) and (2.6) are reduced to an equation:

$$2|\Lambda| = -8\pi Gv^2 \left( \sqrt{B} \frac{dF}{dr} - \frac{n}{r} \sin F \right) \left( \sqrt{B} \frac{dF}{dr} + \frac{n}{r} \sin F \right). \quad (2.33)$$

Since the (anti-)self-dual solitons satisfying Eq. (2.28) make the right-hand side of Eq. (2.33) vanish, we have  $\Lambda = 0$  as a necessary condition for any (anti-)self-dual soliton. Therefore, the static string-like topological and nontopological configurations of  $O(3)$  nonlinear  $\sigma$  model under the static metric (2.2) cannot saturate Bogomolnyi-type bound in (anti-)de Sitter spacetime. In fact static self-dual solitons of this model with a cosmological constant was proved to be constructed only when the metric is stationary and the cosmological constant is negative [17].

In this section we analyzed the  $O(3)$  nonlinear  $\sigma$  model in anti-de Sitter spacetime and found a new static soliton configuration whose nature is nontopological, and its topological charge is a multiple of half integer in addition to the well-known topological lump solution. The obtained solitons are shown to be non-self-dual.

### III. SPACETIME STRUCTURE

We have obtained in the previous section all possible static regular soliton solutions of Eq. (2.4), Eq. (2.5), and Eq. (2.6). In this section we address a question about possible spacetime manifolds formed by  $\sigma$  soliton configurations and a negative vacuum energy. Among the known (2+1)D anti-de Sitter spacetime solutions intriguing ones are regular hyperboloid and BTZ black hole [5,4]. In Ref. [6], one of the authors showed that static global  $U(1)$  vortex can form a space with two event horizons, which resembles a charged BTZ black hole. Specifically, what we are looking for is the existence of black hole horizon, which is manifested by the region of non-positive  $B(r)$ .

At first let us investigate the structure of spatial manifolds by the topological lump solutions and show that any regular topological lump configuration does not form a BTZ-type black hole even when the magnitude of negative cosmological constant is small and the symmetry breaking scale is of the order of the Planck mass. From the asymptotic form of  $B(r)$  in Eq. (2.21), one can easily read a necessary condition to have negative  $B(r)$ . When  $B_\infty$  is not negative, the series expansion (2.21) of  $B(r)$  is always positive for large  $r$  and it implies impossibility of the existence of the horizon. On the other hand, Eq. (2.18) tells an opposite possibility that  $B(r)$  of an  $n = 1$  soliton can have zero at some  $r$ , if  $4\pi G(B_0 + n^2)F_0^2$  is much larger than the magnitude of cosmological constant  $|\Lambda|$ . In order to clarify this issue let us examine the integral equations for  $N(r)$  and  $B(r)$  obtained from Eq. (2.5) and Eq. (2.6):

$$N(r) = -8\pi G \int_r^\infty ds s \left( \frac{dF}{ds} \right)^2, \quad (3.1)$$

$$B(r) = e^{-N(r)} \left\{ 2|\Lambda| \int_0^r ds s e^{N(s)} - 8\pi G v^2 n^2 \int_0^r ds \frac{e^{N(s)}}{s} \sin^2 F + e^{N(0)} \right\}. \quad (3.2)$$

First term in the square bracket of Eq. (3.2) describes contribution of the negative vacuum energy, and second term of it does the core mass. In order to obtain negative  $B(r)$  region for some  $r$ , small magnitude of the negative cosmological constant is favorable. Since the third term  $e^{-N(0)}$  is of order one, another necessary condition from the second term in Eq. (3.2) is the lower bound of symmetry breaking scale  $v$  which must be the Planck mass, i.e.,  $8\pi G v^2 \sim 1$ . To evaluate the value of  $B_\infty$  in Eq. (2.21), we take a crude approximation such as

$$N(r) = 0, \quad (3.3)$$

and

$$F(r) = \begin{cases} 0 & \text{for } 0 < r < r_c - \Delta \\ \pi/2 & \text{for } r_c - \Delta \leq r \leq r_c + \Delta \\ \pi & \text{for } r > r_c + \Delta \end{cases}. \quad (3.4)$$



Inserting Eqs. (3.3) and (3.4) into the integral equation (3.2) and comparing the result with Eq. (2.21), we obtain

$$B_\infty \sim 1 - 16\pi G v^2 n^2 \left( \frac{\Delta}{r_c} \right). \quad (3.5)$$

Since both  $r_c$  and  $\Delta$  have the scale of soliton core size and then the ratio  $\Delta/r_c$  is of the order one, we can confirm that the Planck scale as a symmetry breaking scale is necessary to exhibit the horizon of a BTZ black hole.

Now let us assume that there exists a horizon at  $r_H$ . At each horizon a set of appropriate boundary conditions is

$$B(r_H) = 0, \quad (3.6)$$

$$\left. \frac{dF}{dr} \right|_{r_H} = \frac{\frac{v^2 n^2}{r_H^2} \sin 2F(r_H)}{16\pi G r_H \left( \frac{|\Lambda|}{4\pi G} - \frac{v^2 n^2}{r_H^2} \sin^2 F(r_H) \right)}. \quad (3.7)$$

Since  $B(0) = 1$  and  $B(r) \xrightarrow{r \rightarrow \infty} |\Lambda| r^2$ , the region of negative  $B(r)$  should be bounded and thereby the number of horizons should be even. We attempt a series solution near the horizon  $r_H$  to leading order:

$$F(r) \approx F(r_H) + \frac{\frac{v^2 n^2}{r_H^2} \sin 2F(r_H)}{16\pi G r_H \left( \frac{|\Lambda|}{4\pi G} - \frac{v^2 n^2}{r_H^2} \sin^2 F(r_H) \right)} (r - r_H), \quad (3.8)$$

$$N(r) \approx N(r_H) + \frac{1}{32\pi G r_H} \frac{\left( \frac{v^2 n^2}{r_H^2} \sin 2F(r_H) \right)^2}{\left( \frac{|\Lambda|}{4\pi G} - \frac{v^2 n^2}{r_H^2} \sin^2 F(r_H) \right)^2} (r - r_H), \quad (3.9)$$

$$B(r) \approx 8\pi G r_H \left( \frac{|\Lambda|}{4\pi G} - \frac{v^2 n^2}{r_H^2} \sin^2 F(r_H) \right) (r - r_H). \quad (3.10)$$

Suppose that there exists a region of negative  $B(r)$  bounded by  $r_H^{in}$  and  $r_H^{out}$  ( $r_H^{in} < r < r_H^{out}$ ). Then other necessary conditions are  $\left. \frac{dB}{dr} \right|_{r_H^{in}} < 0$  and  $\left. \frac{dB}{dr} \right|_{r_H^{out}} > 0$ , and they lead to  $\frac{|\Lambda|}{4\pi G} - \frac{v^2 n^2}{(r_H^{in})^2} \sin^2 F(r_H^{in}) < 0$  and  $\frac{|\Lambda|}{4\pi G} - \frac{v^2 n^2}{(r_H^{out})^2} \sin^2 F(r_H^{out}) < 0$  by Eq. (3.10). However, now that  $F(r)$  is monotonically increasing from  $F(0) = 0$  to  $F(\infty) = \pi$ , the negativity of the numerator of the second term in Eq. (3.8) forces a condition to  $F(r)$ , that the value of  $F(r_H^{in})$  should be larger than  $\pi/2$  and that of  $F(r_H^{out})$  should be smaller than  $\pi/2$ . Therefore above conclusion, i.e.,  $F(r_H^{in}) > F(r_H^{out})$ , contradicts to the monotonically increasing property

of  $F(r)$ . Therefore we arrive at a no-go conclusion that axially symmetric regular static topological lump solution in  $O(3)$  nonlinear  $\sigma$  model cannot support a BTZ-type black hole with two horizons in anti-de Sitter spacetime.

Since we have proved that any  $B(r)$  corresponding to regular topological lump configuration cannot be negative, the remaining question for the nonexistence of the black hole horizon is to show the positivity of the minimum of  $B(r)$ . Again, let us assume that there exists a point  $r_H$  such that  $B(r_H) = 0$  and this is the minimum value of  $B$ . Then the position of the horizon  $r_H$  and the value of  $F(r_H)$  are determined in a closed form from Eqs. (2.4) and (2.6):

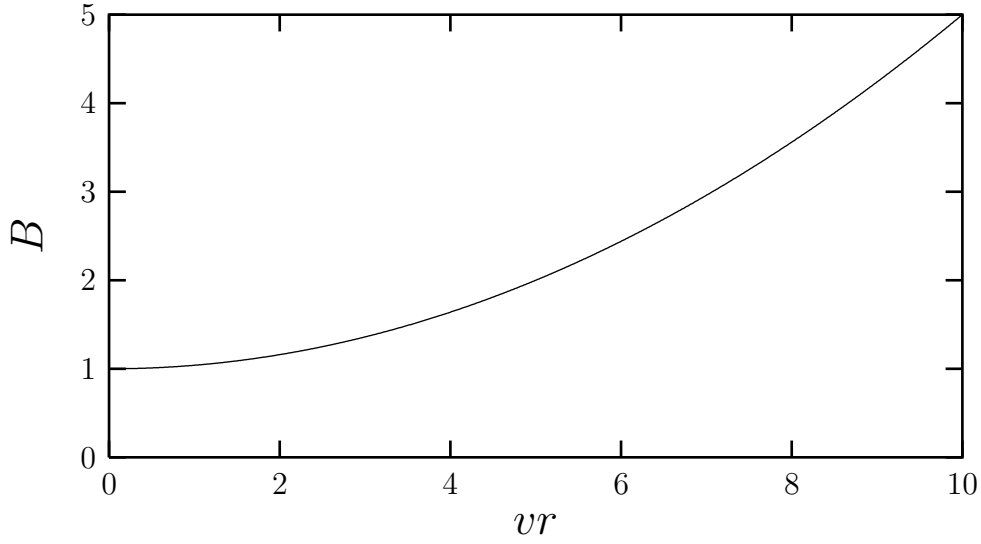
$$r_H = \sqrt{\frac{4\pi G v^2 n^2}{|\Lambda|}} \quad \text{and} \quad F(r_H) = \frac{\pi}{2}. \quad (3.11)$$

If there exists regular solution to have  $B(r_H) = 0$ , one can try a series expansion around the horizon  $r_H$  such as

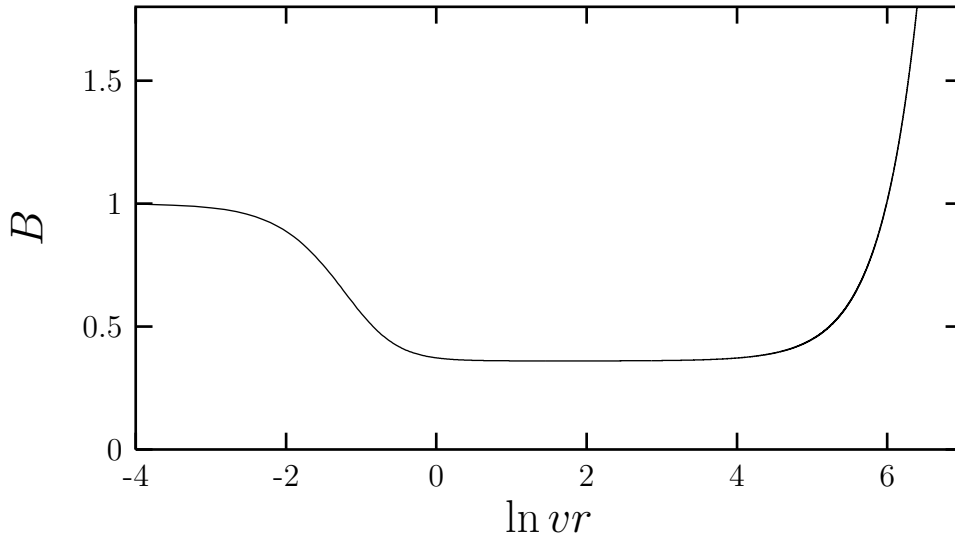
$$F(r) \approx \frac{\pi}{2} + f_1(r - r_H) + f_2(r - r_H)^2 + f_3(r - r_H)^3 + \dots, \quad (3.12)$$

$$B(r) \approx B_2(r - r_H)^2 + B_3(r - r_H)^3 + \dots. \quad (3.13)$$

After replacing  $N(r)$  dependent term in Eq. (2.4) by use of Eq. (2.5), we substitute Eq. (3.12) and Eq. (3.13) into the modified equations (2.4) and (2.6). The comparison of both sides of the equations results in the trivial solution of  $F(r)$ , i.e.,  $0 = f_1 = f_2 = f_3 = \dots$ . It means that the topological lump which is a nontrivial solution cannot constitute spatial manifold of an extremal black hole with one horizon. Combining with the previous proof, we conclude that any regular topological lump of  $O(3)$  nonlinear  $\sigma$  model does not form spacetime of a BTZ black hole irrespective of the values of  $|\Lambda|/v^2$  and  $8\pi G v^2$ . Therefore, the shapes of  $B(r)$  from the regular topological lump solutions are classified into two categories: One is monotonically increasing  $B(r)$  and the other is convex down  $B(r)$  (See Fig. 5).



(a)



(b)

FIG. 5. Two characteristic shapes of  $B(r)$  formed by the topological lumps: (a) A monotonically increasing  $B(r)$  when  $8\pi Gv^2 = 8 \times 10^{-8}$ ,  $|\Lambda|/v^2 = 0.04$ , and  $F_0 = 1250$ , (b) A convex down  $B(r)$  when  $8\pi Gv^2 = 0.2$ ,  $|\Lambda|/v^2 = 4.0 \times 10^{-6}$ , and  $F_0 = 5.896$ .

Behavior of  $B(r)$  given in Fig. 5 describes the structure of the spatial hypersurface of the (2+1)-dimensional spacetime. Since the metric is static, spatial manifold is characterized

by the circumference  $l(r) \equiv 2\pi r$  and the radial distance  $\mathcal{R}(r) = \int_0^r dr/\sqrt{B(r)}$ . We embed it into a three-dimensional hyperbolic space by introducing the third axis  $Z$  such that  $\mathcal{R}^2 = -Z^2 + r^2/B_m$ , where  $Z \geq 0$  and  $B_m$  is the minimum of  $B(r)$ . For sufficiently large  $r$ ,  $B(r) \sim |\Lambda|r^2 + B_\infty$  as given in Eq. (2.21). Introducing variables such as  $\sqrt{|\Lambda|/B_\infty} r = \sinh \chi$  and  $\sqrt{B_\infty} \theta = \Theta$ , we obtain the asymptotic metric

$$ds^2 \approx \frac{1}{|\Lambda|} (d\chi^2 + \sinh^2 \chi d\Theta^2). \quad (3.14)$$

The asymptotic region of two-dimensional spatial manifold given by Eq. (3.14) is a hyperboloid with deficit angle  $2\pi(1 - \sqrt{B_\infty})$ . By use of Eq. (3.5) we estimate the deficit angle to be  $16\pi^2 G v^2 n^2$ . This can easily be understood by the nonexistence of a long tail term in energy-momentum tensor. Since nonvanishing independent components of it are

$$T^t_t = \frac{v^2}{2} B \left( \frac{dF}{dr} \right)^2 + \frac{n^2 v^2}{2r^2} \sin^2 F, \quad (3.15)$$

$$T^r_r = -\frac{v^2}{2} B \left( \frac{dF}{dr} \right)^2 + \frac{n^2 v^2}{2r^2} \sin^2 F, \quad (3.16)$$

they look to include a long tail term. However, substituting Eq. (2.19) into Eqs. (3.15) and (3.16), we read that the leading term is  $\mathcal{O}(1/r^4)$  term which does not affect the asymptotic region of two-dimensional spatial manifold.

As we can expect from Fig. 5, the spatial manifold on the core of topological lump is involved in one of two categories. When the absolute value of negative cosmological constant is large enough, i.e.,  $|\Lambda|/v^2 > 8\pi G F_0^2 \delta_{1n}$  and  $B_m = 1$ , the relation between  $Z$  and  $r$  near the origin is  $dZ \approx \sqrt{\alpha r^2/(1 + \alpha r^2)} dr$  where  $\alpha \equiv |\Lambda| - 8\pi G v^2 F_0^2 \delta_{1n}$ . Then the core region of this soliton is also hyperbolic,  $(Z + 1/\sqrt{\alpha})^2 - r^2 = 1/\alpha$ . On the other hand, when  $B(r)$  is decreasing near the origin, i.e.,  $|\Lambda|/v^2 < 8\pi G F_0^2 \delta_{1n}$  and  $0 < B_m < 1$ , the relation between  $Z$  and  $r'(\equiv r/\sqrt{B_m})$  is given in the following:

$$Z(r) \approx \begin{cases} \sqrt{1 - B_m} r' \left( 1 + \frac{\alpha r'^2}{6(1 - B_m)} \right) & \text{for small } r' \\ \sqrt{\frac{B_\infty}{|\Lambda| B_m} + r'^2} - \sqrt{\frac{B_\infty}{|\Lambda| B_m}} & \text{for large } r', \end{cases} \quad (3.17)$$

and

$$dZ \approx \sqrt{B_{m2}}(r' - r'_m)dr' \quad \text{around } r' = r'_m (\equiv r_m/\sqrt{B_m}), \quad (3.18)$$

where  $B_{m2}$  is the coefficient of the second order term in the series of  $B(r)$  around  $r_m$ . Since  $\alpha$  is negative, the first line in Eq. (3.17) tells us that the core region is convex up. In order to connect smoothly the core and asymptotic regions of the spatial manifold, there should exist an inflection point about the minimum point  $r_m$  of  $B(r)$  as given in Eq. (3.18).

From now on let us look into possible structure of a spacetime manifold formed by the nontopological soliton of half integral winding. Recalling the asymptotic form of  $B(r)$  in Eq. (2.24), one may easily notice a difference between this equation and Eq. (2.21) for the topological lump: The asymptotic space of the half integral winding soliton includes a logarithmic term with negative coefficient. This metric function is the same as that of a global U(1) vortex [6]. In the model of a complex scalar field the very logarithmic term has played a crucial role to constitute a vortex BTZ black hole with two horizons. On the other hand, our nontopological  $\sigma$  solitons are distinguished from global U(1) vortices by the following points. For a given model with fixed model parameters, global U(1) vortex solution is unique, however, there are many nontopological  $\sigma$  soliton solutions characterized by the maximum value of scalar amplitude which is larger than  $\pi/2$  but smaller than  $\pi$ . About the shape of scalar amplitude, the former is a monotonically increasing function from zero to the vacuum expectation value but the latter can contain oscillatory behavior as shown in Fig. 4. Therefore, nontopological  $\sigma$  solitons with the same topological charge are classified into many subclasses by the number of nodes.

The existence of the logarithmic term in the asymptotic form (2.24) of the metric function  $B(r)$  lets us ask an intriguing question about the generation of BTZ black hole for a small magnitude of cosmological constant and relatively large symmetry breaking scale as happened in gravitating global U(1) vortices with a negative cosmological constant. The results of the numerical analysis are summarized in Figs. 4 and 6. Figure 6 shows the metric  $B$  as a function of  $r$  for various number of nodes. As the number of nodes increases (or equivalently the value of  $F_0$  in Eq. (2.16) increases), the value of the minimum of  $B$  decreases. It

is also natural that the behavior of  $B$  is as like as Fig. 6 as the symmetry breaking scale is increased with a fixed value of  $F_0$ . The nontopological  $\sigma$  soliton solutions are seen to tend towards black hole solutions as the symmetry breaking scale  $v$  or the number of nodes is increased, as might be expected. A difference from the behavior of  $B$  for global U(1) vortices can be noticed: In case of the global U(1) vortices, one bump was dug and such minimum of  $B$  finally touched zero value [6], however several bumps are developed for nontopological  $\sigma$  solitons and the outmost one becomes the minimum of  $B$  and then this position tends to be a horizon as shown in Fig. 6. The graphs in Fig. 4 show that wiggles of the scalar field tend to subside to the boundary value  $\pi/2$  outside the location of the minimum of  $B$ .

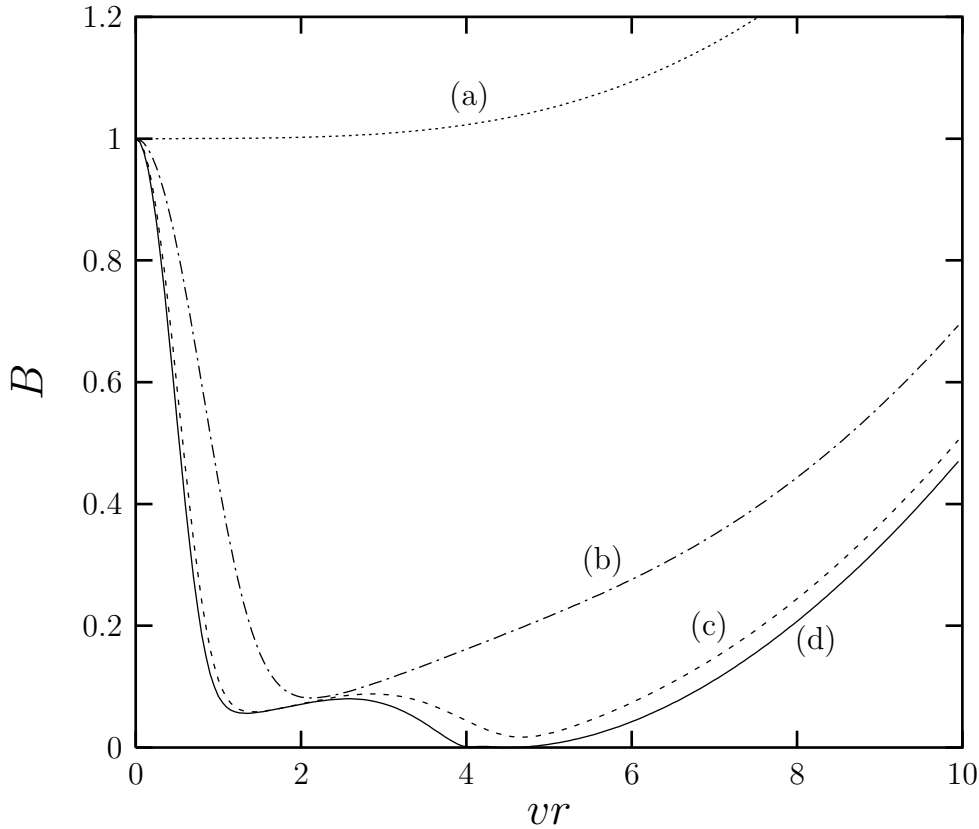


FIG. 6. Plots of  $B(r)$  for various  $F_0$ 's for  $|\Lambda|/v^2 = 0.01$  and  $8\pi Gv^2 = 0.4$ : (a) zero node, (b) one node, (c) two nodes, (d) extremal ( $F_0 = 2.41902$  up to  $10^{-6}$  precision).

Within our numerical precision, a careful analysis of solutions near the transition to a black hole indicates that the nontopological  $\sigma$  soliton loses its scalar amplitude hair as

it develops a horizon. In fact, it is predictable from Eq. (3.8): When  $F(r_H) = \pi/2$ , the actual value of  $dF/dr|_{r_H}$  vanishes for any extremal black hole. Here let us write down the action (2.1) in terms of stereographically projected variables, i.e.,  $\phi^a = v(\sin F \cos(\Theta + \eta), \sin F \sin(\Theta + \eta), \cos F)$ , where the multi-valued  $\Theta$  represents the topological sector and the single-valued function  $\eta$  does the Goldstone degree for a given topological sector. Then, in (2+1)D flat spacetime, we obtain

$$\mathcal{L} = \frac{v^2}{2} \left[ \partial_\mu F \partial^\mu F + \sin^2 F \partial_\mu (\Theta + \eta) \partial^\mu (\Theta + \eta) \right]. \quad (3.19)$$

By use of a duality transformation in 2+1 dimensions [20], one can easily show in the context of the path integral formulation that the above theory (3.19) is equivalent to that of a U(1) vector field  $A_\mu$ :

$$\mathcal{L} = \frac{v^2}{2} \partial_\mu F \partial^\mu F - \frac{1}{4} \frac{F_{\mu\nu} F^{\mu\nu}}{\sin^2 F} + \frac{v}{2} \epsilon^{\mu\nu\rho} F_{\mu\nu} \partial_\rho \Theta, \quad (3.20)$$

where  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ . If the scalar amplitude is frozen to be  $F = \pi/2$ , outside the black hole horizon, then the matter field action (3.20) is nothing but the sum of the Maxwell term and the minimal interaction between the gauge field and point particle. Now we understand the reason why a  $\sigma$  soliton black hole looks just like a charged BTZ black hole outside the horizon [4]. Therefore, nontopological  $\sigma$  solitons in O(3) nonlinear  $\sigma$  model do not break no-hair theorem. This phenomenon seems universal for our nontopological soliton solutions since it happens for a wide range of the symmetry breaking scale  $v$  and the negative cosmological constant  $\Lambda$ . In this aspect, the regular nontopological  $\sigma$  solitons are also distinctive from the topological global U(1) vortices with scalar hair [6], but resemble the case of regular gravitating magnetic monopoles in 3+1 dimensions [12]. We can imitate the case of exact singular monopole solution whose metric is the Reissner-Nordström black hole [21]. Specifically,  $F(r) = \pi/2$ ,  $\Theta = n\theta$ , and  $\eta = 0$ , everywhere and the corresponding black hole spacetime is a charged BTZ-type. More plausible singular configurations may be obtained by changing the boundary condition of the metric function at the origin, i.e.,  $B(0) \neq 1$  similar to the monopole black hole [12]. Since the singularity of the fields which is

presumably at the origin can be hidden behind a horizon, we may not exclude the possibility that singular solutions can form small BTZ black holes lying within a nontopological  $\sigma$  soliton. Since no non-Abelian scalar hair can penetrate the horizon for regular solitons, we can evaluate the position of the horizon by using Eqs. (2.4) and (2.6), and it is nothing but the formula (3.11). The values of the horizon obtained by numerical analysis coincide with those from Eq. (3.11) within precision.

As mentioned previously, we have many nontopological soliton excitations classified by the number of nodes for a given topological sector of the theory so that we have to discuss stability among these classical solutions carrying with the same topological charge. A good method is to compare their masses. Since the obtained spacetime is not asymptotically flat but is hyperbolic, the usual Arnowitt-Deser-Misner mass is not obtained in the limit  $r \rightarrow \infty$ . For the energy per unit length of infinitely-long axially symmetric systems, known expressions are the C-energy [22] and the conserved quasilocal mass [23]. Here we use the latter of which expression for the static observer is given by

$$M_q \equiv \frac{1}{4G} \sqrt{e^{2N(r)} B(r)} \left( \sqrt{|\Lambda| r^2 + 1} - \sqrt{B(r)} \right), \quad (3.21)$$

$$\xrightarrow{r \rightarrow \infty} \begin{cases} 2\pi n^2 v^2 \left( \frac{\Delta}{r_{core}} \right) & \text{for the topological lump} \\ \pi n^2 v^2 \left[ \ln \left( \frac{r}{r_{core}} \right) + 2 \sin^2 \beta \left( \frac{\Delta}{r_{core}} \right) \right] & \text{for the nontopological soliton,} \end{cases} \quad (3.22)$$

where Eq. (2.21) was used in the right-hand side of the above expression. The mass for the topological lump has only the constant term. It is obvious because this lump is localized around its core without a long range tail term. The nontopological soliton of half integral winding has logarithmically divergent mass term in addition to the core mass. It shows some resemblance between static global U(1) vortex and the nontopological soliton in O(3) nonlinear  $\sigma$  model, whose leading long range term is the same, i.e.,  $T_t^t \sim 1/r^2$  for large  $r$ .

For the  $n = 1$  class of solutions we compare the values of the quasilocal mass (3.21) of no node solution, one node solution, two node solution, the solution of an extremal black hole, charged BTZ black hole at a sufficiently large distance  $vr = 50$  as a function of  $v$  with fixed  $8\pi G = 0.4$  and  $|\Lambda| = 0.01$  (See Table 1). The tendency that the quasilocal mass increases



for higher node solutions looks universal, and further numerical studies for various  $G$ ,  $|\Lambda|$ , and  $v$  also keep the same behavior. Therefore, no node solution is the lowest energy solitonic excitation among those with a given charge  $n/2$ . Since (2+1)D Einstein gravity does not have any attractive propagating gravitational degree, it seems natural. All half integral winding solitons are nontopological, so excited spectra may decay into the no node soliton of the lowest energy. This procedure may presumably be correct for the solitons in the space of a regular hyperboloid because the system has massless Goldstone degrees. Now, if we recall that no node solution with the monotonically increasing  $F(r)$  cannot form a black hole horizon, then an intriguing question is raised about the stability of an extremal BTZ-type black hole. In 3+1 dimensions, attractive gravitational force usually makes a matter distribution with mass larger than the critical value unstable and leads to the gravitational collapse where the destination is the formation of a black hole. It seems unlikely for our  $O(3)$  nonlinear  $\sigma$  model in (2+1)D anti-de Sitter spacetime. On the other hand, there may be an opposite procedure that an extremal BTZ-type black hole is produced but it is energetically unfavorable and then the horizon disappears. However, we need further study on the stability of nontopological solitons to settle down this issue. Now a comment about the critical symmetry breaking scale is in order. In any natural environment the magnitude of negative cosmological constant is much lower than the symmetry breaking scale  $v$ , and the very symmetry breaking scale  $v$  is much lower than the Plank scale. For example, if we consider a present universe with an extremely small bound of the negative cosmological constant ( $|\Lambda| \sim 10^{-83}\text{Gev}^2$ ), the critical value of the symmetry breaking not to form a BTZ-type black string is about  $10^{-2}\text{eV}$  which is very low energy. Of course, the above estimation is far from realistic situation before we take into account the anisotropy in cosmic ray background and other cosmological fluctuations.

node	0	1	2	extremal
$v = 1$	0.01245	0.01918	0.02056	0.02080
$v = 1.5$	0.02144	0.02625	0.02646	0.02647
$v = 2$	0.02664	0.02840	0.02841	0.02841

Table 1. The values of quasilocal mass of various node solutions and the extremal charged BTZ black hole at a large distance  $vr = 50$  with  $8\pi G = 0.4$  and  $|\Lambda| = 0.01$ .

#### IV. GEODESIC MOTIONS

The study of time-like and null geodesics is an adequate way to visualize the form of interaction on the soliton and the feature of its spacetime. Let us analyze possible geodesic motions and clarify whether a test particle experiences attraction or repulsion due to the soliton. The geometry depicted by Eq. (2.2) admits the rotational killing vector  $\partial/\partial\theta$  and the static killing vector  $\partial/\partial t$ , so two corresponding constants of motion along geodesics are

$$\gamma = Be^{2N}\frac{dt}{ds} \quad \text{and} \quad L = r^2\frac{d\theta}{ds}, \quad (4.1)$$

where  $s$  is an affine parameter along the geodesic. Since the space is not asymptotically flat, the constant  $\gamma$  cannot be interpreted as the local energy of the test particle at infinity. The radial geodesic equation is

$$\frac{1}{2}\left(\frac{dr}{ds}\right)^2 = -\frac{1}{2}\left[B(r)\left(m^2 + \frac{L^2}{r^2}\right) - \frac{\gamma^2}{e^{2N(r)}}\right] = -V(r), \quad (4.2)$$

where the mass of the test particle can be rescaled as  $m = 1$  for time-like geodesics and  $m = 0$  for null geodesics. We analyze the trajectories of test particles for the topological lump background and the nontopological soliton background separately, and they are divided into four categories according to whether they have mass ( $m = 1$ ) or not ( $m = 0$ ), or whether their motions are purely radial ( $L = 0$ ) or rotating ( $L \neq 0$ ). As shown in Fig. 5 and Fig. 6, the geometry of spatial manifolds of our  $\sigma$  model solitons is similar to those of global U(1) vortices [6]. Here we briefly mention different points.

## IV.1 Topological Soliton

The main character of the spacetime structure of topological lumps is the absence of black hole. Due to this character, the geodesic motions are simple. It is qualitatively similar to the regular hyperboloids by global  $U(1)$  vortices [6].

For the radial motion ( $L = 0$ ) of a massless test particle ( $m = 0$ ),  $B(r)$  dependence disappears in the effective potential  $V(r)$ . The allowed motions are (i) stopped particle motion for  $\gamma = 0$  and (ii) unbounded motion for  $\gamma \neq 0$  with the speed  $dr/ds = \gamma/\sqrt{2}$  at spatial infinity. Since  $N(r)$  is monotonically increasing, this massless test particle in a radial motion always feels attractive force.

For the rotational motions ( $L \neq 0$ ) of a massless test particle ( $m = 0$ ), the effective potential includes the centrifugal force term  $L^2 B(r)/2r^2$  which forbids the test particle to access the soliton core. Therefore, any allowed rotational motion should have the minimum value of radius  $r_{min}$  that  $r \geq r_{min}$ . Since the value of the effective potential is  $(|\Lambda|L^2 - \gamma^2)/2$  at spatial infinity, any allowed motion should be bounded by the minimum radius  $r_{min}$  and the maximum radius  $r_{max}$  when  $|\Lambda|L^2 > \gamma^2$ . However we cannot see this easily due to the smallness of  $|\Lambda|$ . When  $|\Lambda|L^2 \leq \gamma^2$ , the motions are also divided into two classes by the peak speed: One is the class with the peak speed at infinity, and the other is that with the peak speed at a finite radius.

The effective potential for the radial motion ( $L = 0$ ) of a massive test particle ( $m = 1$ ) is

$$V(r) = \frac{1}{2} \left( B(r) - \frac{\gamma^2}{e^{2N(r)}} \right). \quad (4.3)$$

For large  $r$ , it is approximated as

$$V(r) \approx \frac{|\Lambda|}{2} r^2 + \frac{1}{2} (B_\infty - \gamma^2) + \mathcal{O}(1/r^2), \quad (4.4)$$

and then all possible motions are bounded. Since the power series expansion of  $V(r)$  for small  $r$  is

$$V(r) \approx \frac{1}{2}(1 - \gamma^2 e^{-2N(0)}) + \left[ \left( \frac{1}{2}|\Lambda| - 4\pi G v^2 F_0^2 (1 - \gamma^2 e^{-2N(0)}) \right) r^2 \right] + \dots, \quad (4.5)$$

we divide the shapes of the potential (4.3) into two classes. When the negative vacuum energy dominates the repulsive force of the scalar field even at the core of the soliton, i.e.,  $|\Lambda|/2 - 4\pi G v^2 F_0^2 (1 - \gamma^2 e^{-2N(0)}) \geq 0$ ,  $V(r)$  is monotonically increasing and thereby the force is attractive everywhere. Then the minimum of the effective potential is at the origin and its value is  $|\Lambda|/16\pi G v^2 F_0^2$ . The leading constant term in Eq. (4.3), which is the minimum of  $V(r)$ , tells us that the radial motions are allowed only when  $\gamma \geq e^{N(0)}$ . On the other hand, when  $|\Lambda|/2 - 4\pi G v^2 F_0^2 (1 - \gamma^2 e^{-2N(0)}) \leq 0$ , the test particle with  $\gamma$  smaller than the critical value  $\gamma_{cr}$  ( $\gamma_{cr} = e^{N(0)} \sqrt{1 - |\Lambda|/16\pi G v^2 F_0^2}$ ) feels repulsive force at the core of the soliton. The allowed value of  $V(0)$  lies between  $|\Lambda|/16\pi G v^2 F_0^2$  and  $1/2$ . One may expect that there exists the negative region of  $V(r)$  between  $r_{min}$  and  $r_{max}$ , however our numerical work shows the absence of such region. Possible motions are (i) the stopped motion, (ii) the oscillation between the minimum radius and the maximum radius, (iii) rolling to the origin, as  $\gamma$  decreases.

For the circular motions ( $L \neq 0$ ) of a massive test particle ( $m = 1$ ), the effective potential takes general form (See Eq. (4.2)). Since the centrifugal force term dominates at small  $r$ ,  $V(r)$  for small  $r$  resembles that of the case of a rotating motion of a massless test particle, and there exists perihelion  $r_{min}$ . For large  $r$ , all motions are bounded by an aphelion  $r_{max}$  because of the negative cosmological constant term. The allowed motions are (i) the circular orbit at  $r_{circ}$  when  $\gamma = \gamma_{circ}$ , and (ii) the bounded orbit between perihelion  $r_{min}$  and aphelion  $r_{max}$  when  $\gamma$  is larger than  $\gamma_{circ}$ . Noticing the vanishment of  $V(r)$  at both  $r_{min}$  and  $r_{max}$ , one may suspect that the comoving time defined by

$$\tau = \int dr \frac{\gamma}{\sqrt{-2V(r)}}, \quad (4.6)$$

diverges when the test particle approaches to those points. However, since the denominator in Eq. (4.6) is proportional to  $1/\sqrt{r - r_{min}}$  (or  $1/\sqrt{r_{max} - r}$ ), it takes finite comoving time to reach a boundary. So does the coordinate time defined by  $dt/d\tau = \gamma/Be^{-2N}$  since there is no black hole horizon, i.e.,  $B(r) > 0$  for all  $r$ .

## II.2 Nontopological Soliton

As we have discussed in the previous section, there exist black hole solutions for some nontopological solitons. For some regular solutions, e.g., (a) and (b) in Fig. 6, the geodesic motions are not so much different from those of topological solitons. There are different  $B$ 's with several bumps as shown in the graphs (c) and (d) in Fig. 6. One may suspect that these  $B$ 's generate different geodesic motions, e.g., two isolated radial regions in the effective potential  $V(r)$ . However, our numerical works show that there are no such effective potential, so that the character of geodesic motions for regular nontopological solitons is the same as that for topological lumps. The only difference is the rapid variation of  $V(r)$  near the origin, due to rapidly increasing  $N(r)$ . Note that Eq. (2.5) reflects the rapid increasing of  $N(r)$  for many nodes of our nontopological soliton.

From Eqs. (4.1) and (4.2), the elapsed coordinate time  $t$  of a test particle which moves from  $r_0$  to  $r$  is

$$t = \int_r^{r_0} \frac{dr}{B(r)e^{N(r)}\sqrt{1 - \frac{1}{\gamma^2}(m^2 + \frac{L^2}{r^2})B(r)e^{2N(r)}}}. \quad (4.7)$$

It diverges when the test particle approaches to a point where  $B(r)$  vanishes at least linearly. As we expected, the spacetime with horizons depicts that of a black hole. For the black hole solutions, our geodesic motions outside the horizon are intrinsically the same with that of a charged BTZ black hole, since any scalar hair does not penetrate the horizon but the logarithmic Goldstone sector.

As usual, the matter distribution is reflected to the scalar curvature which is given by

$$R = -6\Lambda - 16\pi GT^\mu_\mu. \quad (4.8)$$

For small  $r$ , Eq. (4.8) for both the topological lump and the nontopological soliton becomes

$$R \approx 6|\Lambda| - 8\pi Gn^2v^2F_0^2[2 + (|\Lambda| - 8\pi Gv^2F_0^2\delta_{1,n})r^2]r^{2n-2}. \quad (4.9)$$

When  $n = 1$ , the curvature can be negative due to the accumulation of the matter at the core of the soliton at the Planck scale. For large  $r$ , the behavior of the scalar curvature depends on the characteristic of the solitons:

$$R \approx \begin{cases} 6|\Lambda| - 32\pi G v^2 F_\infty^2 |\Lambda| \frac{1}{r^4} & \text{for the topological lump} \\ 6|\Lambda| - 8\pi G v^2 n^2 \frac{1}{r^2} - 32\pi G v^2 |\Lambda| F_{\pi/2,\infty}^2 \frac{1}{r^4} & \text{for the nontopological soliton.} \end{cases} \quad (4.10)$$

As expected, the space is curved at large  $r$  for the nontopological soliton, while it is not for the topological lump. Although we have charged BTZ black holes from some half integral winding soliton configurations, we may expect that all the obtained spacetimes do not contain physical curvature singularity due to the regularity of the matter fields and the metric functions everywhere. It is easily checked by the Kretschmann scalar,

$$\begin{aligned} R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} &= 4G_{\mu\nu}G^{\mu\nu} \\ &= 4\text{Tr}\left[\text{diag}\left(-\frac{1}{2r}\frac{dB}{dr}, -\frac{1}{2r}\frac{dB}{dr} - \frac{B}{r}\frac{dN}{dr}, -\frac{1}{2}\frac{d^2B}{dr^2} - \frac{3}{2}\frac{dB}{dr}\frac{dN}{dr} - B\frac{d^2N}{dr^2} - B\left(\frac{dN}{dr}\right)^2\right)\right]. \end{aligned} \quad (4.11)$$

When both  $N(r)$  and  $B(r)$  are regular everywhere, the only possible singularity can be at the origin in Eq. (4.11), however it is also regular at the origin due to the behaviors of those metric functions at the origin as given in Eqs. (2.17) and (2.18). Then, the spacetime formed by the topological lump or the nontopological soliton is always regular everywhere irrespective of the existence of the black hole horizon.

## V. CONCLUSION AND DISCUSSION

In this paper we have studied static soliton solutions of  $O(3)$  nonlinear  $\sigma$  model coupled to Einstein gravity with a negative cosmological constant. It has been shown that any regular static soliton configuration with axially symmetric static metric is not self-dual in this anti-de Sitter spacetime. By examining second order Euler-Lagrange equations, we obtained a new class of nontopological soliton solutions whose winding number is multiple of half integer in addition to the well-known topological lumps with integral topological charge. Scalar amplitude of the topological lump solution is monotonically increasing function which interpolates the symmetric vacuum and the broken vacua, and its energy density, the time-time component of energy-momentum tensor, is localized around the soliton core. The lack of a long tail term in the energy density at asymptotic region leads to nonexistence of a

BTZ-type black hole irrespective of symmetry breaking scale. The only spatial structure formed by the topological lump is regular hyperboloid with deficit angle.

On the other hand, the asymptotic behavior of the nontopological solitons shows oscillation around its boundary value  $\pi/2$ , and these solutions are characterized by the number of nodes for a given parameter set of the model. The energy expressions of these nontopological solitons include a logarithmic term at asymptotic region, and this property resembles that of global U(1) vortices. According to the scale of the negative cosmological constant, we obtained the following spacetimes: One of them is regular hyperboloid with deficit angle and the other is charged BTZ black hole. The conserved quasilocal mass of the BTZ black hole is composed of two terms, i.e., one of them is finite core mass and the other is logarithmically divergent term.

Here we have several comments on some resemblance and difference between our half integral winding  $\sigma$  solitons and the global U(1) vortices. First, the former solutions are nontopological, but the latter solutions are topological. Therefore, the energetics of our nontopological solitons should be checked to confirm their stability, which may provide a clue to distinguish one from the other. Second, the global U(1) vortex is unique regular soliton configuration with monotonically increasing scalar amplitude for a given set of model parameters. On the other hand, a number of nontopological solitons exist in a given model, which are characterized by the number of oscillations in scalar amplitude. Third, both solitons carry a long range term ( $\sim 1/r^2$ ) in the expressions of their energy density due to nontrivial phase winding sector of Goldstone modes. The solutions have been seen to tend towards black holes as the symmetry breaking scale increases and the magnitude of negative cosmological constant becomes small. The black hole generated by a nontopological  $\sigma$  soliton is a charged BTZ black hole without non-Abelian scalar hair, while a small BTZ black hole lying within a global U(1) vortex is available where nontrivial scalar field exists outside the horizon.

Since the Einstein gravity in 2+1 dimensions does not have propagating degrees of freedom, the introduction of a negative vacuum energy plays a drastic role for making the soliton

excitations rich in scalar theories. It made the global U(1) vortices free from the physical curvature singularity in the model of a spontaneously broken global U(1) symmetry. In our O(3) nonlinear  $\sigma$  model this attractive force supports the nontopological solitons, which have never been obtained without adding a gauge field and explicit symmetry breaking scalar potential [24] except for some unstable, spherically symmetric solitons in (3+1)D de Sitter spacetime [25]. The obtained spacetimes include charged BTZ black hole. In this context it may also be intriguing to ask the same question to local vortices in Abelian Higgs model [26,27]. When we consider the stability of the obtained solutions or general straight infinite cosmic strings, various forms of metric can also be taken into account, e.g., a metric with boost invariance along the string direction,  $ds^2 = e^{2N(r)}B(r)(dt^2 - dz^2) - \frac{dr^2}{B(z)} - r^2d\theta^2$ , or the general form of static metric,  $ds^2 = e^{2N(r)}B(r)(dt - C(r)dz)^2 - \frac{dr^2}{B(z)} - r^2d\theta^2 - D(r)dz^2$ , or even a stationary one,  $ds^2 = e^{2N(r)}B(r)(dt - E(r)r d\theta)^2 - \frac{dr^2}{B(z)} - r^2d\theta^2$ .

Throughout this paper we have considered the cases where the deficit angle is smaller than  $2\pi$ . If we recall that supermassive local vortices produced various geometrical structures including an analog of Kasner spacetime, a cylinder, or a two sphere [27,28], we may expect some drastic change of (anti-de Sitter) spacetime formed by the topological lumps in the Planck scale. In relation with time-dependent soliton configurations, once stationary Q-lump solution is generated and forms a black hole structure [29], it must be a spinning black hole in 2+1 dimensions.

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